

Ozobot Evo Classroom Application: Discovering Pascal's Triangle

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Introduction

One of the most famous of all triangles is **Pascal's triangle**, named after the French mathematician Blaise Pascal. The triangle is formed by placing 1's at the end of each row, and then obtaining the rest of the numbers in the triangle by adding the two nearest numbers above each number. Figure 1 shows the first six rows of Pascal's triangle. It is traditional to number the rows starting with 0 at the apex of the triangle. The two short arrows in Figure 1 show, as an example, how the number 10 is obtained by adding 4 and 6 from the previous row.

Row 0				1				Row Sum = $1 = 2^0$
Row 1			1		1			Row Sum = $2 = 2^1$
Row 2		1		2		1		Row Sum = $4 = 2^2$
Row 3		1		3		3		Row Sum = $8 = 2^3$
Row 4		1		4		6		Row Sum = $16 = 2^4$
Row 5	1		5		10		10	Row Sum = $32 = 2^5$

Figure 1

There are many interesting properties of Pascal's triangle. For example, the triangle is symmetric with respect to a vertical line drawn through the apex of the triangle. As shown at the far right of Figure 1, the sum of the numbers in each row is a power of 2, with the general pattern of 2^n for row n . The Ozobot Evo experiment that you will do in this classroom application makes use of this fact. The long arrow in Figure 1 points to a diagonal that contains the so-called *triangular numbers*.

Interfacing Ozobot Evo and Pascal's Triangle

For reference purposes while discussing how to interface Ozobot Evo with Pascal's triangle, Figure 2 shows a small image of the OzoMap used in this classroom application. A larger version, intended for student use, can be found on the last page of this document and duplicated for student lab groups.

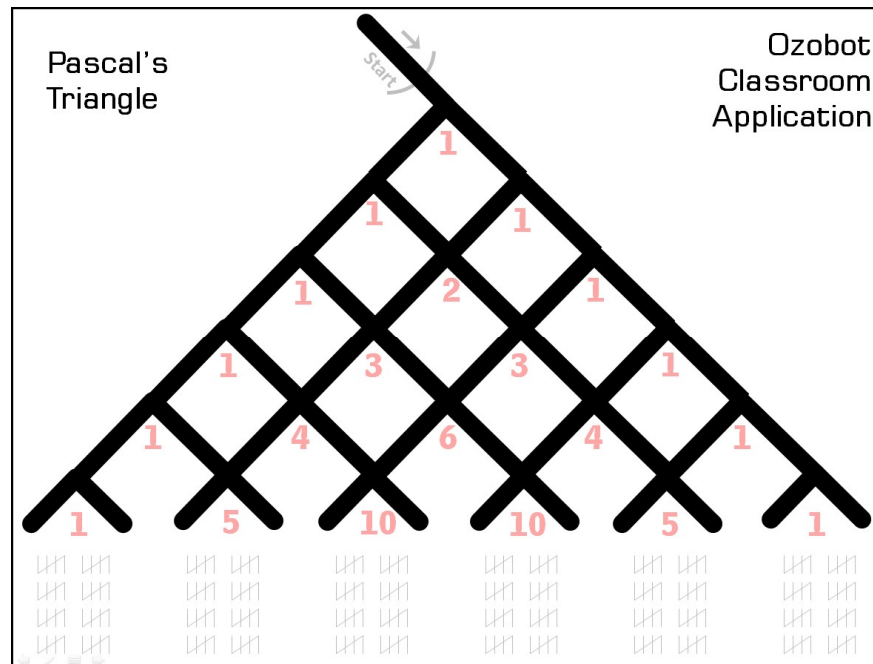


Figure 2

What we need is a configuration providing paths for Ozobot Evo to take on Pascal's triangle. Ozobot Evo starts at the apex of the triangle and then takes random paths through the triangle until he reaches the bottom and stops. We program him so that **at every intersection he is allowed to move only in the downward direction**. At each intersection he randomly chooses to go straight or make a 90° turn, continuing a downward path.

Discovering Pascal's Triangle

In this classroom application, we want to **begin by letting students discover Pascal's triangle before actually working with Ozobot**. To accomplish this, make a copy of the triangle in Figure 3 on page 3 for each lab group. Explain the rule that they start at the location labeled "Start", and can only move downward. Then ask the lab groups to:

1. Determine the values of A, B, and C by finding the number of **different paths** from the start to points A, B, and C. They should find that A, B, and C are all 1. They should record the values on the triangle.
2. Ask them to do the same for points D, E, and F. They should find that D=1, E=2, and F=1, recording the values on the triangle.
3. Ask them to do the same for points G, H, I, and J. They should find that G=1, H=3, I=3, and J=1, recording the values on the triangle.
4. Ask them to do the same for points K, L, M, N, and O. This should be a bit more challenging, but if they are careful, they should find that K=1, L=4, M=6, N=4, and O=1, again recording the values on the triangle.
5. Now ask them to see if they can determine the values of P, Q, R, S, T, and U, simply from considering the values in the previous row. At this point many students, by observing patterns, will notice that (a) there are always 1's on the far left and far right of each row, and (b) internal numbers can be found by

adding the two nearest numbers in the previous row. The values of P through U should be P=1, Q=5, R=10, S=10, T=5, and U=1.

- Next, ask the students to add up the numbers in each row. They should find that the row sums double with each successive row—1, 2, 4, 8, 16, and 32, going up by powers of 2. ***The student's have discovered Pascal's triangle!***

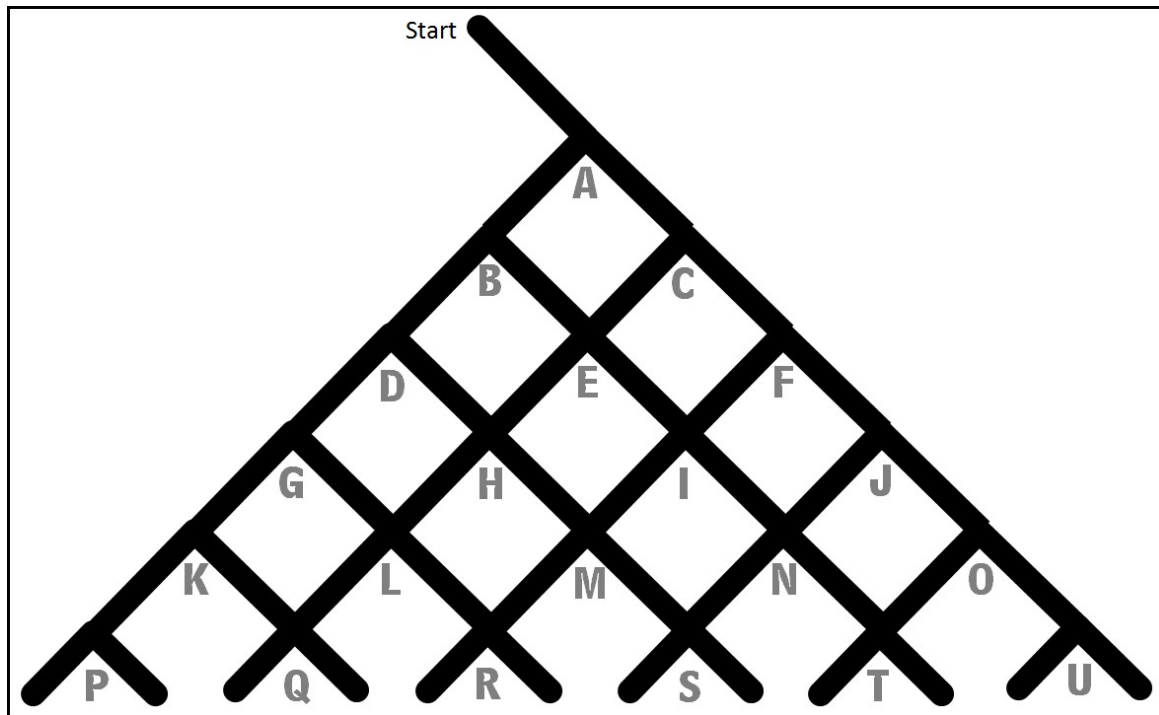


Figure 3

Enter Ozobot Evo

Now you can distribute the OzoMaps and Ozobot Evos to the lab groups. Remind them that Ozobot Evo has been programmed to move only downward on Pascal's triangle and ***randomly decides at each intersection to go straight or turn, each with a probability of ½.***

- Ask the students to compute the theoretical probabilities that Ozobot Evo will wind up stopping at each of the points P, Q, R, S, T, and U. *(The total number of different equally likely paths to these points is the sum $1 + 5 + 10 + 10 + 5 + 1 = 32$. For example, the theoretical probability of Ozobot reaching point S is $10/32 = 0.3125$.)*
- Now the lab groups should be ready to do an experiment in probability. If you have not loaded the ozocode program in advance, have the students load the program *DiscoveringPascalsTriangleWithEvo.ozocode* (share code is 2j9uxc). Ask each group to place Ozobot on the location labeled *start* with the leading edge of Ozobot on the short curved gray line. They should turn Evo on and then start the program by double-pressing the start button. Evo will show its red LED while it is moving. Each time Evo reaches the bottom and stops, the LED will turn green for five seconds, allowing time to move Evo back to the start location at the apex of Pascal's Triangle. They should mark the appropriate tally mark each time Evo reaches the bottom and stops. They can

record tallies for as much time as time permits, hopefully from ten to fifteen minutes for each group, though less time is OK if you have a fair number of lab groups. They should be able to record five to six tallies per minute of data collection. See Figure 4 for a sample of what the tally sheet might look like when completed by a lab group. The numbers at the bottom of Figure 4 are the tallies recorded as whole numbers.

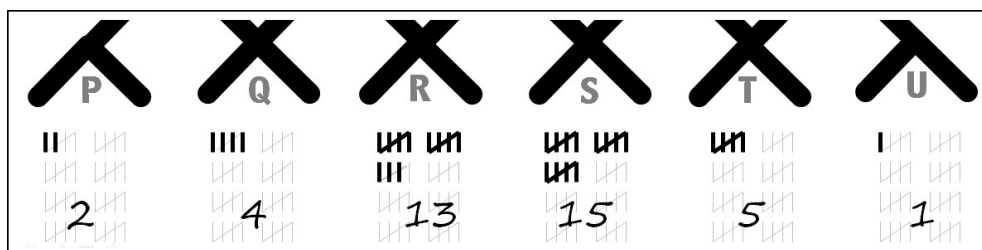


Figure 4

Lab groups should notice that a pattern is emerging, though it may not be clear with the small sample size of a single group.

Data Analysis

Page 5 contains a data table in which the data from all lab groups can be combined for a more detailed data analysis. In particular, we will be interested in comparing the theoretical probabilities for Ozobot stopping at each of the points P through U to the empirical probabilities when combining all of the lab groups' data. The larger sample size obtained by combining all lab groups should make the pattern much clearer.

The **grand total** is the sum of the six column totals. The **empirical probability** is obtained by the formula:

$$\text{Empirical Probability} = \frac{\text{Column Total}}{\text{Grand Total}}$$

The **theoretical probabilities** are obtained by dividing the numbers 1, 5, 10, 10, 5, and 1 by their sum, 32, as was discussed on page 3.

The Percent Difference can then be obtained by the formula:

$$\text{Percent Difference} = \frac{\text{Empirical Probability} - \text{Theoretical Probability}}{\text{Theoretical Probability}} \times 100\%$$

If the empirical probability is less than the theoretical probability, the percent difference will be negative. If the reverse is true, then the percent difference will be positive.

Typical class results are shown in the table on page 6.

Pascal’s Triangle: Lab Group Analysis Data Table



Lab Group	P	Q	R	S	T	U	
							Grand Total↓
Total →							
Empirical Probability							
Theoretical Probability							
Percent Difference							

For the Teacher: Typical Classroom Results



Lab Group	P	Q	R	S	T	U	
1	2	4	13	15	5	1	
2	1	7	14	13	9	0	
3	3	10	18	18	5	1	
4	3	11	11	16	10	2	
5	3	10	11	21	3	1	
6	0	9	9	10	10	3	
7	0	4	14	13	10	2	
8	3	11	15	13	4	1	
9	3	5	19	12	6	1	
10	1	5	12	14	5	0	
							Grand Total↓
Total →	19	76	136	145	67	12	455
Empirical Probability	0.042	0.167	0.299	0.319	0.147	0.026	
Theoretical Probability	0.031	0.156	0.313	0.313	0.156	0.031	
Percent Difference	35.5%	7.1%	-4.5%	1.9%	-5.8%	-16.1%	

Discussion Questions

1. Why do you think the percent differences are greatest on the left and right, and generally decrease toward the center of the triangle?
2. What would be the values of the numbers in the n=6 row of Pascal's triangle? (Remember that the first row is row zero.) [Answer: 1, 6, 15, 20, 15, 6, 1]

Preparation for the Lesson

1. Make a copy of the OzoMap on the last page of this document for each lab group.
2. Make a copy of Figure 3 on page 3 for each lab group.
3. Make a copy of the data table on page 5 for each group.
4. If you want to save class time, load the program *DiscoveringPascalsTriangleWithEvo.ozocode* (**share code is 2j9uxc**) into your Ozobot Evos prior to the class meeting.
5. The students should turn Evo on and then start the program by **double-pressing** the start buttons when the light turns blue. Ozobot will show a red LED while moving on the OzoMap. It will show a green LED after each traversal of the triangle, during which the students can move Ozobot Evo back to the start.

More Pascal Triangle Discoveries (Optional)

1. **(Binomial Expansions)** Ask your students to expand each of the following binomials, with terms in the order of decreasing powers of x . Ask them what they notice about the coefficients of the terms in the expansions.
 - $(x + 1)^0$
 - $(x + 1)^1$
 - $(x + 1)^2$
 - $(x + 1)^3$
 - $(x + 1)^4$
2. **(Binomial Expansions)** Just by using Pascal's triangle, ask the students to expand $(x + 1)^6$.
3. **(Combinatorics)** You have a paper clip, pencil, and an eraser on the table. If order is not important:
 - In how many ways can you grab none of the items on the table?
 - In how many ways can you grab exactly one of the items on the table?
 - In how many ways can you grab exactly two items on the table?
 - In how many ways can you grab all three of the items on the table?

What do you notice about your answers to these four questions? *[Here the students are investigating the number of combinations of n things taken k at a time (often called " n choose k "), where $n=3$ and $k=0, 1, 2$, and 3 . This is commonly denoted in a number of ways, including: $C(n, k)$, nC_k , ${}_nC_k$, and $\binom{n}{k}$. The number in the n 'th row of Pascal's triangle and k 'th position of that row, both n and k starting a zero, is $\binom{n}{k}$.]*

4. **(Combinatorics)** You have a bag of 6 different marbles. In how many ways can you select 4 marbles from the bag, if order of selection is not important? $\left[\binom{6}{4} = 15.\right]$
5. **(Probability)** If you toss a penny, nickel, dime and quarter, what is the probability of obtaining 0 tails? 1 tail? 2 tails? 3 tails? 4 tails? *[Have the students list all possible PNDQ permutations:*

0 Tails $\rightarrow$ HHHH

1 Tails $\rightarrow$ HHHT, HHTH, HTHH, THHH

2 Tails $\rightarrow$ HHTT, HTHT, HTTH, THHT, THTH, TTHH

3 Tails $\rightarrow$ HTTT, THTT, TTHT, TTTH

4 Tails $\rightarrow$ TTTT

Have them observe that there are 1, 4, 6, 4, and 1 independent possible outcomes, respectively, for tossing 0, 1, 2, 3, and 4 tails. This gives a total of $1 + 4 + 6 + 4 + 1 = 16$ independent, equally probable outcomes. Therefore, the probability for tossing 0 tails is $1/16 = 0.0625$, for tossing 1 tail is $4/16 = \frac{1}{4} = 0.25$, etc. Again, we see Pascal's triangle at work!]

7. If you toss 6 coins, what is the probability of obtaining exactly 3 heads and 3 tails? [$\binom{6}{3}/2^6 = 20/64 = 0.3125$.]

Ozobot Classroom Application

